Joint Frequency and Angle Estimation Algorithms

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***Abstract*— This paper presents subspace decomposition based algorithms for joint frequency of arrival (FOA) and direction of arrival (DOA) estimation. We present three different methods named QR-TLS, LU-TLS and SVD-TLS to tackle the problem of joint FOA and DOA estimation for multiple incident source signals. The type of unitary transformation method used, differentiate between the three methods. The method receives a snapshot of the incident source signal and forms a Toeplitz structured data matrix from the received signal. Three different unitary transformation methods called LU, QR and SVD are applied to extract the signal and noise subspaces from the data matrix. The joint FOA and DOA estimates are obtained from the signal subspace by applying total least squares (TLS) method. The proposed method has several advantages Firstly, the proposed method employs only a single snapshot of the received data to construct a Toeplitz structured matrix. The rank is directly related to both FOA and DOA of the incident source signals independent of the signals being coherent or non-coherent; therefore, an algorithm does not require any spatial smoothing. Second, the proposed method avoids costly operations of computing singular value decomposition (SVD) or eigenvalue decomposition (EVD) of the covariance matrix and hence it can provide rapid FOA and DOA estimates. The simulation results are presented that compares the proposed method performance in terms of accuracy and processing time of three factorization techniques and estimation methods for both coherent and non-coherent source signals.**

***Index Terms*— Joint DOA & FOA Estimation, QR-TLS, LU-TLS, SVD-TLS**

I. Introduction

ANTENNA array processing techniques have been extensively been utilized for direction of arrival estimation (DOA) of the incident source signals [1-7] Estimating the incident source signal frequency jointly with the direction of arrival is of great importance in an application of radar and wireless signal processing. These applications include source localization and estimating carrier frequency estimation at the receiving end of a wireless communication system. Maximum likelihood based optimal estimation techniques are applicable but are computationally intensive [8]. ESPRIT based joint angle and frequency estimation methods have been proposed in [9]. Multi resolution ESPRIT is used for joint angle frequency estimation in [10]. The ESPRIT based have successfully been applied in several areas including radar based localization system and space division multiple access (SDMA) [9], [11]. Another iterative tri-linear decomposition method for joint angle and frequency estimation has been proposed [12]. A similar but more simplistic method for joint angle frequency estimation called JAFE is proposed in [13]. The drawback of the ESPRIT based methods is that they require EVD or SVD of the cross spectral data matrix of the received data which needs high computational complexity.

The joint FOA and DOA estimation problem for multiple incident source signals on an antenna array becomes challenging in the presence of highly correlated and coherent sources. One of the most popular methods of performing joint FOA and DOA estimation is the JAFE algorithm proposed in [14]. Unfortunately its main limitation is that it can perform poorly in the presence of coherent signals. To overcome this problem, a preprocessing method called spatio-temporal smoothing (STS) can be used is proposed in [15] based on constructing a spatio-temporally smoothed data matrix before performing the JAFE algorithm. However, both of these methods involve extra pre-processing computations. Recently, we have proposed a QR-TLS ESPRIT method for joint DOA and FOA estimation of multiple incident source signals [16] . This method employs less costly QR factorization method as compared to the eigenvalue decomposition methods and work equally well for both coherent and non-coherent signal sources.

In this paper, we propose a subspace decomposition based algorithm that provides fast and reliable joint angle and frequency estimates. The underlying ideas behind the proposed method are as follows. Firstly, the proposed method maps the single snapshot of the received data into a Toeplitz structured data matrix whose rank is related to both the FOA and DOA of the incident sources independently with respect to

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sources being coherent or non-coherent. This gives the proposed method advantage of not requiring any spatial smoothing techniques, resulting in a reduction of computational complexity and cost. The Toeplitz structure of the problem also reduces the computational cost in determining signal space from the data matrix. The complexity of finding the QR decomposition of a Toeplitz matrix is *O(nlogn)* in contrast to the $O(n^2)$ for other matrices [17]. Secondly, the proposed method requires only a single snapshot of data to produce the joint FOA and DOA estimates, thus favoring real time implementation. Thirdly, the integration of the TLS method provides more accurate estimates in the presence of noisy measurements. Finally, the proposed method requires a small snapshot length of the received signal to estimate the FOA and DOA jointly. The low implementation complexity, computational cost, and reliable performance of the proposed method favors its real time applications. The method is named QR-TLS ESPRIT since it employs ESPRIT algorithm in conjugation with QR and TLS methods.

The paper is organized as follows: Section II develops the system model for joint FOA and DOA estimation. The proposed methods QR-TLS, LU-TLS and SVD-TLS for joint FOA and DOA estimation are detailed in Section III. In Section IV the simulation results are presented and compared with some variations in the same method. Finally, we conclude our paper in Section V.

II. System model

The system model assumes K sources lying in a far-field region of a uniform linear array (ULA) composed of $2N + 1$ elements as shown in Figure 1. The center element of the array is considered as the reference element. Each source has a carrier frequency f_i and is assumed to be lying at an angle θ_i with reference to the ULA where $i = 1,2,3, \ldots K$ as shown in Figure 2.

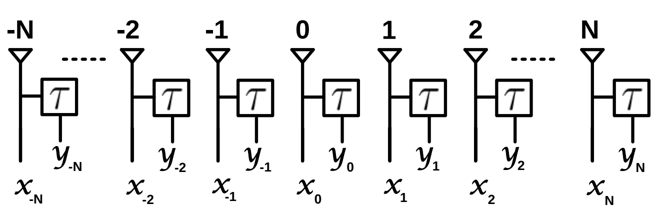

Figure 1: Uniform linear array configuration composed of 2N + 1 elements.

The signal received at the m^{th} element of ULA is given as [1]

$$x_m(t) = \sum_{i=1}^{K} \left[e^{\frac{j2\pi m d f_i \sin\theta_i}{c}} s_i(t) + n_i(t) \right] \tag{1}$$

In (1), where x_m is the received signal at the m^{th} element of the array, θ_i is angle of the i^{th} source from the array reference, f_i is the frequency of the i^{th} source element, d is inter-sensor spacing, c is the speed with which the electromagnetic wave propagates and n_i is additive white Gaussian noise with $\mu = 0$ and $\sigma = 1$ ($\square(0,1)$). The exponent term in (1)

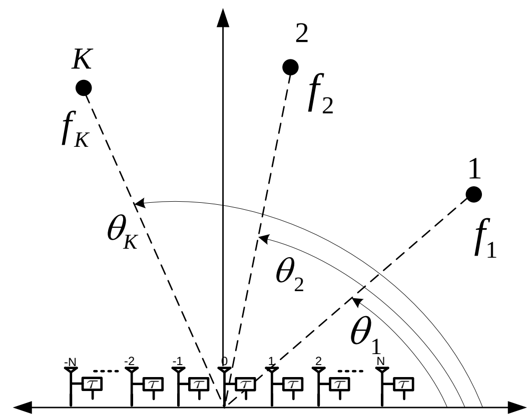

Figure 2: K sources lying in the far-field of a ULA

represents the phase shift the signal undergoes relative to the signal received at the reference element and s_i denotes the i^{th} signal signal for $i = 1, 2, \cdots, K$.

In proposed method we construct $(N+1)\times 1$ sub arrays where each sub array consists of $(N+1)\times 1$ elements. An observation matrix is constructed from a single snapshot of data received at time t from each of the $N+1$ sub arrays. The $(N+1)\times(N+1)$ dimension observation data matrix X is formulated as

$$X = \begin{bmatrix} x_0 & x_{-1} & \cdots & x_{-N} \\ x_1 & x_0 & \cdots & x_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_N & x_{N-1} & \cdots & x_0 \end{bmatrix} \tag{2}$$

$$X = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_N \end{bmatrix} \tag{3}$$

where $\gamma_i = A(\theta, f)\Phi(\theta, f)^i S + n_i$, for $i = 0, 1, \cdots, N$. The S is the source vector and $A(\theta, f)$ is the array factor of the ULA and $\Phi(\theta, f)$ is a diagonal phase delay matrix given as

$$A(\theta, f) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j2\pi\frac{df_1 \sin\theta_1}{c}} & e^{j2\pi\frac{df_2 \sin\theta_2}{c}} & \cdots & e^{j2\pi\frac{df_K \sin\theta_K}{c}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j2\pi\frac{Ndf_1 \sin\theta_1}{c}} & e^{j2\pi\frac{Ndf_2 \sin\theta_2}{c}} & \cdots & e^{j2\pi\frac{Ndf_K \sin\theta_K}{c}} \end{bmatrix} \tag{4}$$

$$\Phi(\theta, f) = diag\begin{pmatrix} e^{-j2\pi\frac{df_1 \sin\theta_1}{c}} & e^{-j2\pi\frac{df_2 \sin\theta_2}{c}} & \cdots & e^{-j2\pi\frac{df_K \sin\theta_K}{c}} \end{pmatrix} \quad (5)$$

In order to estimate the frequencies, single delayed outputs of the received signal at the antenna array from K sources are added as shown in Figure 1. The delayed received signal becomes

$$\begin{aligned} y_m(t-\tau) &= \sum_{i=1}^{K}\left[e^{j2\pi\frac{mdf_i \sin\theta_i}{c}} s_i(t-\tau) + n_i(t) \right] \\ &= \sum_{i=1}^{K}\left[e^{j2\pi\frac{mdf_i \sin\theta_i}{c}} s_i(t) e^{-j2\pi f_i \tau} + n_i(t) \right] \end{aligned} \quad (6)$$

where y_m is the delayed version of the signal received signal at the m^{th} element of the array.

The $(N+1)\times(N+1)$ dimension delayed observation data matrix Y is given as

$$Y = \begin{bmatrix} y_0 & y_{-1} & \cdots & y_{-N} \\ y_1 & y_0 & \cdots & y_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_N & y_{N-1} & \cdots & y_0 \end{bmatrix} \quad (7)$$

$$Y = \begin{bmatrix} \Omega_0 & \Omega_1 & \cdots & \Omega_N \end{bmatrix} \quad (8)$$

where, $\Omega_i = A(\theta, f)\phi(\theta, f)^i \Psi(f) S + n_i$, for $i = 0, 1, \cdots, N$

$$\Psi(f) = diag\left(e^{-j\beta_1} \quad e^{-j\beta_2} \quad \cdots \quad e^{-j\beta_K} \right) \quad (9)$$

where $\beta_k = 2\pi f_k \tau$ for $k = 1, 2, \cdots, K$

The observation matrices X and Y are grouped into a matrix W of dimension $2(N+1)\times(N+1)$ which will be utilized in joint FOA and DOA estimation in the following section.

$$W = \begin{bmatrix} X \\ Y \end{bmatrix} \quad (10)$$

III. Proposed Joint FOA & DOA Estimation Methods

A. *Proposed QR-TLS ESPRIT Method*

The proposed method is based on subspace decomposition. The method exploits the fact that the necessary signal subspace and the noise subspace of a signal is inherent in the observation matrix 'W' which can be extracted by applying QR decomposition method [18], [19]. In the following subsection, we describe briefly the steps for our proposed QR-TLS ESPRIT method takes in order to estimate the frequencies and angles of the multiple incident sources.

Frequency Estimation

The rotation matrix $\Psi(f)$ which contains the information about the frequencies of the multiple incident sources can be found by applying the straight forward least squares (LS) approach [20]. However, under noisy measurements in practical situations the LS may become inappropriate method to apply. In such case $\Psi(f)$ can be solved using the TLS method [21].

The following steps are taken in order to estimate the multiple incident source frequencies.

1. In the first step, the data matrix W is decomposed into $2(N+1)\times 2(N+1)$ orthogonal matrix Q and a $2(N+1)\times(N+1)$ upper triangular matrix R using QR factorization.

$$\mathrm{W} = QR \quad (11)$$

2. The signal space of the data matrix W is obtained by selecting the first K columns of Q which forms an ortho-normal basis for the signal subspace of W [22].

$$Q_s = [q(1), q(2), ..., q(K)] \quad (12)$$

where $q(i)$ denotes the i^{th} column of Q.

3. In order to estimate the source frequencies, the rotation matrix $\Psi(\mathrm{f})$ is determined by partitioning the $2(N+1)\times K$ matrix Q_s into two sub-matrices Q_{s1} and Q_{s2} where each sub-matrix is of $(N+1)\times K$ dimension.

$$Q_s = \begin{bmatrix} Q_{s1} \\ Q_{s2} \end{bmatrix} \quad (13)$$

The $\Psi(f)$ matrix is determined by applying Total Least Squares (TLS) solution as follows:

I. Apply the QR decomposition on the matrix formed as

$$[U, V] = qr(Q_s^H Q_s^T) = qr\left(\begin{bmatrix} Q_{s1}^H \\ Q_{s2}^H \end{bmatrix} \begin{bmatrix} Q_{s1} & Q_{s2} \end{bmatrix} \right) \quad (14)$$

where U, V corresponds to Q and R of the QR factorized results respectively.

II. Partition U into $K\times K$ sub-matrices such that

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \quad (15)$$

III. Compute the eigenvalues (λ's) of the matrix Ψ given as

$$\Psi = -U_{12}U_{22}^{-1} \quad (16)$$

4. Estimate the source frequencies from the eigenvalues obtained in Step 3.

$$\hat{f}_k = \frac{\angle\lambda_k}{2\pi\tau} \quad (17)$$

where $\hat{f}_k$ is the estimated frequency of the k^{th} source for $k = 1, 2, \cdots, K$.

Angle Estimation

1. In order to estimate the angles of the incident source signals we construct two sub-matrices from Q_{s1} with a phase shift as follows

$$Q_1 = Q_{s1}(1 : N-1, :) \quad (18)$$

$$Q_2 = Q_{s1}(2 : N, :) \quad (19)$$

2. The source angles are estimated from the phase delay matrix Φ which can be determined by applying TLS as follows:

I. Apply the QR decomposition on the matrix formed as

$$[U_1, V_1] = qr(Q_{s1}^H Q_{s1}) = qr\left(\begin{bmatrix} Q_1^H \\ Q_2^H \end{bmatrix}\begin{bmatrix} Q_1 & Q_2 \end{bmatrix}\right) \quad (20)$$

where U_1, V_1 corresponds to Q and R of the QR factorized results respectively.

II. Partition U_1 into $K \times K$ sub-matrices such that

$$U_1 = \begin{bmatrix} U_{1,11} & U_{1,12} \\ U_{1,21} & U_{1,22} \end{bmatrix} \quad (21)$$

III. Compute the eigenvalues (λ's) of the matrix Φ given as

$$\Phi = -U_{1,12}U_{1,22}^{-1} \quad (22)$$

IV. Estimate the elevation angle from the eigenvalues obtained in the last step and the earlier estimated frequencies using the following expression

V.

$$\hat{\theta}_k = \sin^{-1}\left(\frac{\angle\lambda_k c}{2\pi d\hat{f}_k}\right) \quad (23)$$

where $\hat{\theta}_k$ is the estimated DOA of the k^{th} source for $k = 1, 2, \cdots, K$.

B. Proposed LU-TLS ESPRIT Method

A variant of the proposed QR-TLS ESPRIT method called LU-TLS ESPRIT method is presented in this section. LU decomposition is a useful technique to solve system of linear equations. Inherently it is based on Gaussian elimination used here to decompose signal and noise subspaces. The rest of the method follows similar procedure as discussed earlier for the frequencies and angles estimation of the multiple incident sources.

Frequency Estimation

1. In the first step, the data matrix W is decomposed into $2(N+1)\times 2(N+1)$ lower triangular matrix L and a $2(N+1)\times(N+1)$ upper triangular matrix U using LU factorization.

$$\mathrm{W} = LU \quad (24)$$

The first K columns of the data matrix W are selected which corresponds to the signal subspace of W.

$$L_s = [l(1), l(2), ..., l(K)] \quad (25)$$

where $l(i)$ denotes the i^{th} column of L.

2. The rotation matrix $\Psi(\mathrm{f})$ is determined by partitioning the $2(N+1)\times K$ matrix L_s into two sub-matrices L_{s1} and L_{s2} where each sub-matrix is of $(N+1)\times K$ dimension.

$$L_s = \begin{bmatrix} L_{s1} \\ L_{s2} \end{bmatrix} \quad (26)$$

The estimates of the incident source signal frequencies are obtained from the rotation matrix $\Psi(f)$ by following the same procedure as discussed earlier in proposed QR-TLS method.

Angle Estimation

1. In order to estimate the angles of the incident source signals, we construct two sub-matrices from L_{s1} with a phase shift as follows

$$L_1 = L_{s1}(1 : N-1, :) \quad (27)$$

$$L_2 = L_{s1}(2 : N, :) \quad (28)$$

2. The rest of the steps to estimate angles of the incident signal angles from the matrix Φ follow the same procedure as described earlier in the Section III-A.

C. Proposed SVD-TLS ESPRIT Method

As the names suggests the proposed SVD-TLS ESPRIT method employs SVD technique for subspace decomposition. In this method, the signal space of the data matrix W is obtained by selecting the first K columns of the matrix S obtained after the SVD decomposition. The rest of the method follows the similar procedure for frequency and angle of arrival estimation as discussed in QR-TLS and LU-TLS

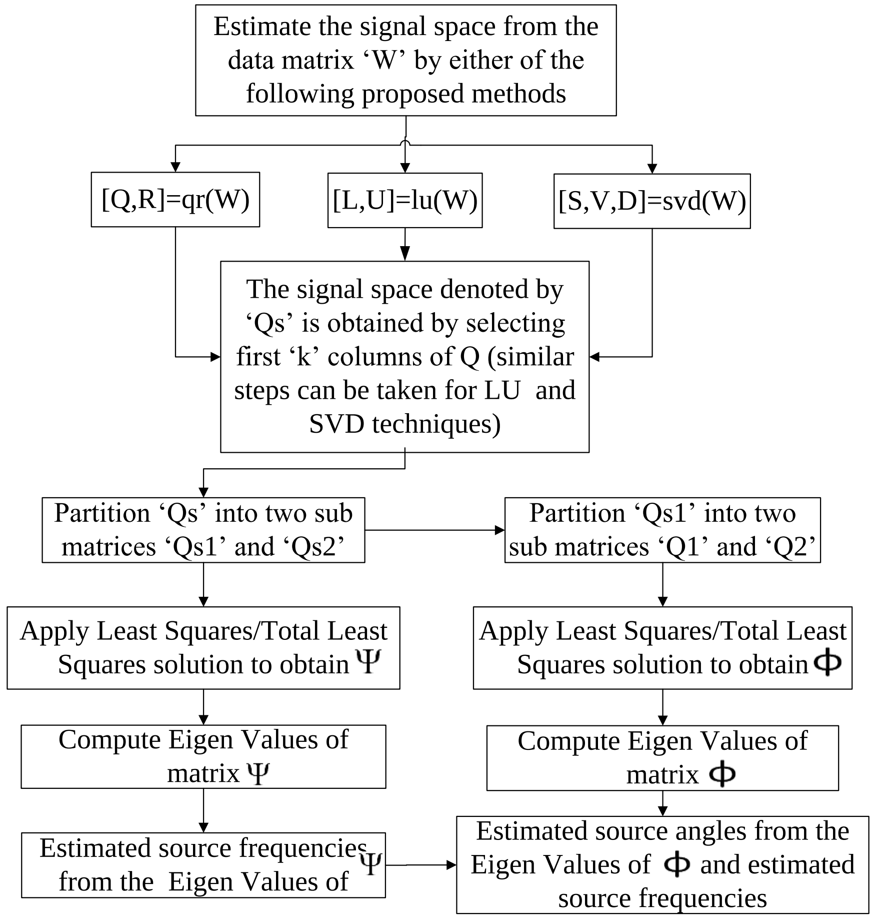

Figure 3: Flow Chart of the proposed QR-TLS ESPRIT, LU-TLS ESPRIT and SVD-TLS ESPRIT methods

methods, thus the details of the method are not included here in order to avoid redundancy. The flow chart shown in Figure 2 summarizes the proposed method procedure for joint FOA and DOA estimation.

IV. Simulation Results

The performance of the proposed method is assessed in simulations with both QR and LU decomposition methods and benchmarked with the conventional SVD technique. Furthermore, the estimates with QR and LU methods are compared when integrated with both TLS and LS. In the following we construct a following test scenario for simulation purpose:

- Number of sensor elements of a ULA: 11
- Number of source elements: 3
- Source frequencies: 500 kHz, 700 kHz and 900 kHz.
- Source angles (from the array reference): 10^o, 20^o, and 30^o.
- Additive white Gaussian noise with mean, $\mu = 0$ and variance $\sigma = 1$.

The SNR of the incoming signal is varied from 0 to 50 dB in steps of 5 dB. For each value of SNR a Monte Carlo simulation with 100 iterations is performed and root mean square error (RMSE) in the estimated source frequencies and directions for the three sources at (500 kHz, 10^o), (700 kHz, 20^o), (900 kHz, 30^o) is calculated from the following expression

$$RMSE = \sqrt{\frac{1}{100}\sum_{m=1}^{100}[a_m - a_0]^2} \qquad (29)$$

where a_m, a_0 denotes the estimated frequency/angle and actual frequency/angle respectively.

The simulations are performed for both coherent and non-coherent sources. Snapshot variable *sp* represents number of consecutive samples taken from each antenna array elements at given time. In each case two different snapshot lengths of $sp = 2$ and $sp = 10$ are considered where a single snapshot of data consists of $2N+1$ samples acquired from $2N+1$

elements of a ULA. Figure 44 and Figure 45 shows the RMSE against the SNR plot for the frequency and the angle estimates respectively for non-coherent signal sources value with $sp = 2$ respectively. The figure shows closely following RMSE curves with the proposed QR-TLS, LU-TLS and SVD methods for the frequency estimates. The figure also depicts less estimation errors of the proposed method when integrated with TLS as compared to the LS. The RMSE curves for the angle estimates for each SNR shows that the estimation errors in DOA estimates with QR-TLS and LU-TLS methods are increased when compared with the SVD method but the magnitude of the errors are low. The SVD method is inherently robust even for nearly singular matrices or highly correlated data matrices. Therefore, performance of SVD methods is comparatively better compare to peer QR and LU decomposition methods. Though, estimation robustness is coming with computational complexity. This fact is apparent from both frequency and angle of arrival estimation results in Figure 3 and Figure 4. It can also be seen from the figure the RMSE of the angle estimates increases with LS approach. The RMSE curves in the frequency and angle estimates for the snapshot length of $sp = 10$ are shown in Figure 46 and Figure 47 respectively. Comparing these curves with the ones obtained for $sp = 2$ case, an improvement in both the frequency and angle estimates is obtained as expected.

In order to assess the performance of the proposed method in case of coherent sources, three correlated source signals are constructed. The correlation coefficient between the first and the second source signal and the first and the third source signals are set to 0.5 and 0.8 respectively. The simulation is performed for snapshot lengths of $sp = 2$ and $sp = 10$. Figure 48 and Figure 49 shows the plot of the RMSE in the frequency and the angle estimates respectively for coherent source signals value with $sp = 2$. Figure 7 shows performance gap of SVD-TLS, QR-TLS, and LU-TLS is narrower compared with broadly uncorrelated case. As expected, this graph also shows better frequency estimates with the proposed method QR-TLS and LU-TLS as compared to the SVD. The figure also shows low RMSE values with the TLS as compared to the LS approach. The RMSE curves for the angle estimates for each SNR shows similar trend as obtained earlier for the non-coherent sources. Again, SVD performance is better compared with other two methods as SVD handles data correlation much better with increased computational load. The estimation errors in DOA estimates with QR-TLS and LU-TLS methods are increased when compared with the SVD method but the magnitude of such error is still very low. The figure also shows increased RMSE in the angles estimates obtained with LS. The RMSE curves in the frequency and angle estimates for the snapshot length of $sp = 10$ are shown in Figure 410 and Figure 4 respectively. An expected, an improvement in both the frequency and angle estimates is obtained with improved data set.

The computational times of the proposed joint FOA and DOA estimation method with QR and SVD decomposition methods integrated with TLS and LS is listed in Table 1. From the table it can be observed that the proposed methods QR-TLS and LU-TLS ESPRIT methods are both computationally less expensive as compared with the SVD ESPRIT. This computational advantage observed from the fact that SVD require complexity in order of $O(mn^2 + n^3)$ while QR decomposition with householder transformation is in order of $O(mn^2 - n^3/3)$. The integration of a TLS method with LU and QR further cuts the computational time and hence favors fast performance of the proposed methods.

V. CONCLUSION

The results of the paper shows fast and reliable performance of the proposed QR-TLS, LU-TLS and SVD-TLS ESPRIT methods for joint FOA and DOA estimation. The proposed method employs only a single snapshot of the received data for the construction Toeplitz data matrix. This data matrix formulation favors joint estimation of the FOA and DOA independent of the signals being coherent or non-coherent. Therefore, the algorithm does not require any spatial smoothing. The proposed method avoids costly operations of computing singular value decomposition (SVD) or eigenvalue decomposition (EVD) of the signal covariance matrix and hence provides rapid FOA and DOA estimates. The simulation results are demonstrates the better performance of the QR-TLS and LU-TLS method as compared to the conventional SVD approach in terms of processing time. In conclusion, the low computational cost, implementation complexity and small snapshot length requirement of the proposed method for joint FOA and DOA estimation makes it a favorable candidate for real time applications.

Table 1: COMPUTATION TIME COMPARISON FOR 100 MONTE CARLO ITERATIONS

Estimation Method	*Computation Time for sp=2 (ms)*	*Computation Time for sp=10 (ms)*
QR-TLS ESPRIT	4.378969	16.8681
QR-LS ESPRIT	5.344397	20.9871
LU-TLS ESPRIT	4.235493	15.7472
LU-LS ESPRIT	5.072982	20.4245
SVD-TLS ESPRIT	5.752679	22.8842
SVD- LS ESPRIT	6.024013	24.5234

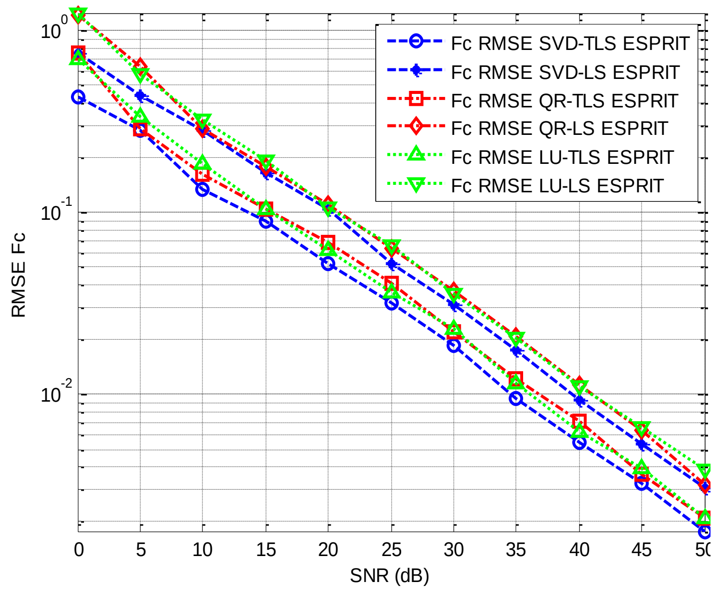

Figure 4: RMSE in frequency estimation using proposed algorithm for three non-coherent sources at (500 KHz, 10^o), (700 KHz, 20^o), (900 KHz, 30^o) and sp=2

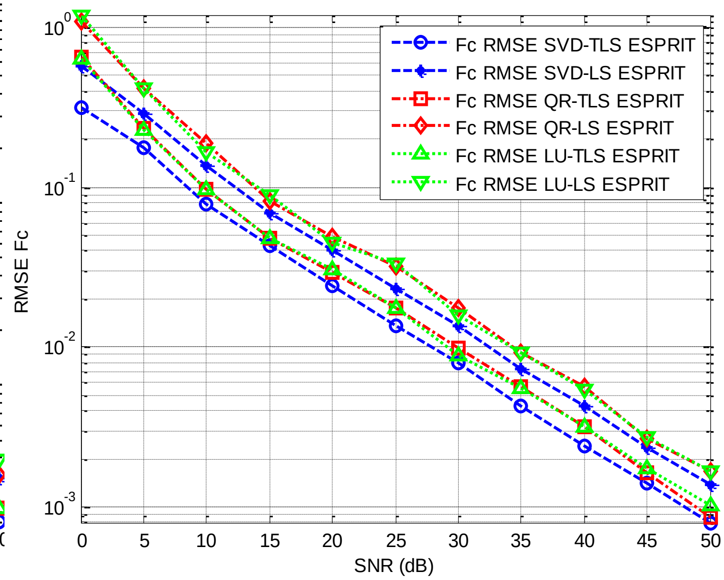

Figure 6: RMSE in frequency estimation using proposed algorithm for three non-coherent sources at (500 KHz, 10^o), (700 KHz, 20^o), (900 KHz, 30^o) and sp=10

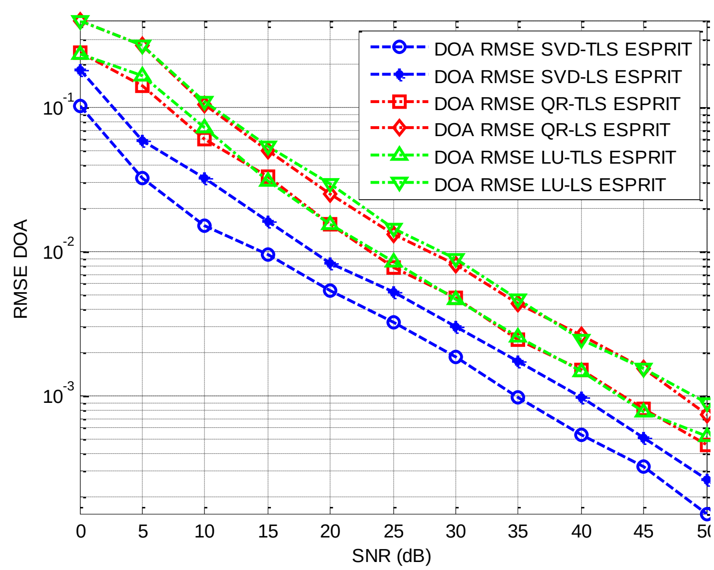

Figure 5: RMSE in angle estimation using proposed algorithm for three non-coherent sources at (500 KHz, 10^o), (700 KHz, 20^o), (900 KHz, 30^o) and sp=2

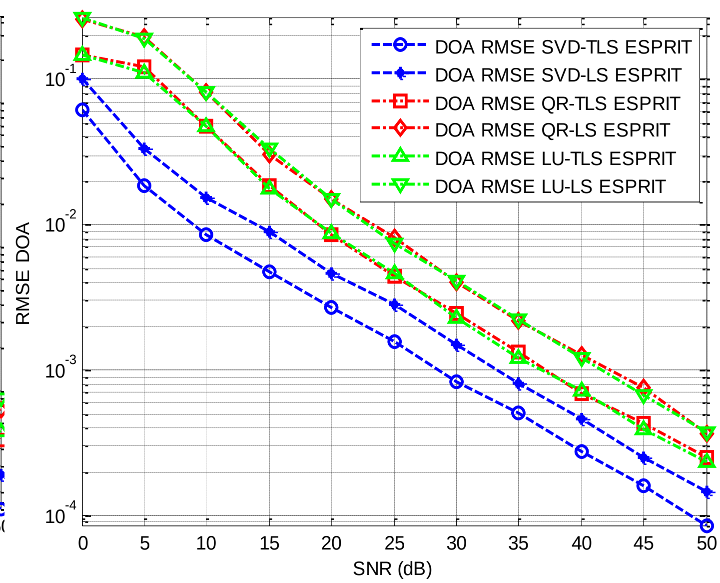

Figure 7: RMSE in angle estimation using proposed algorithm for three non-coherent sources at (500 KHz, 10^o), (700 KHz, 20^o), (900 KHz, 30^o) and sp=10

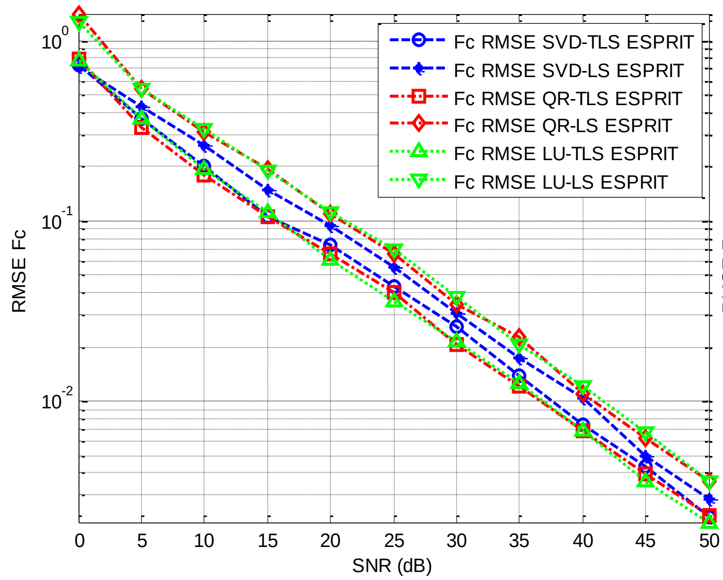

Figure 8: RMSE in frequency estimation using proposed algorithm for three coherent sources at (500 KHz, 10^o), (700 KHz, 20^o), (900 KHz, 30^o) and sp=2

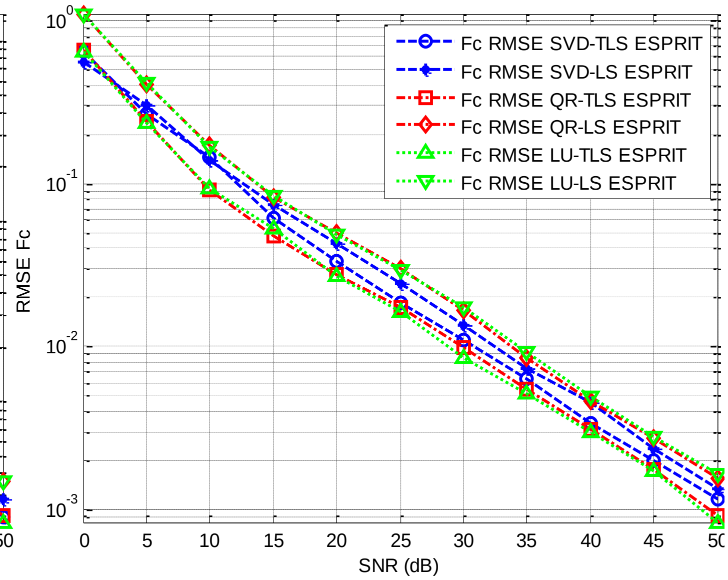

Figure 10: RMSE in frequency estimation using proposed algorithm for three coherent sources at (500 KHz, 10^o), (700 KHz, 20^o), (900 KHz, 30^o) and sp=10

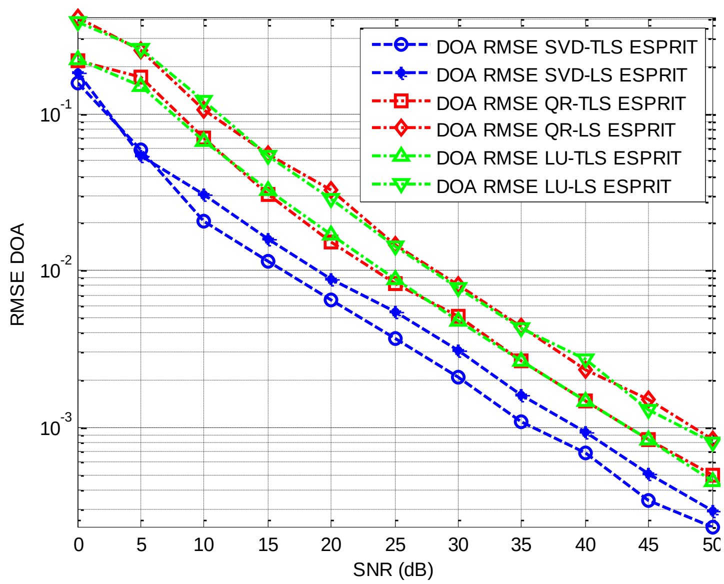

Figure 9: RMSE in angle estimation using proposed algorithm for three coherent sources at (500 KHz, 10^o), (700 KHz, 20^o), (900 KHz, 30^o) and sp=2

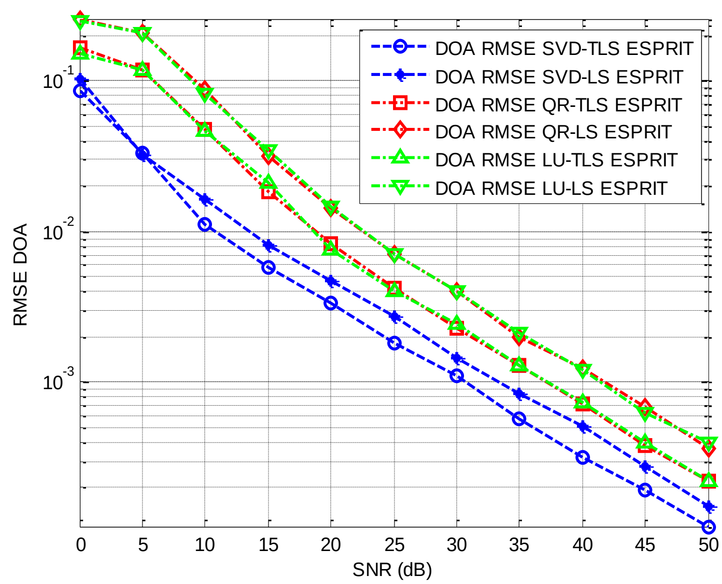

Figure 11: RMSE in angle estimation using proposed algorithm for three coherent sources at (500 KHz, 10^o), (700 KHz, 20^o), (900 KHz, 30^o) and sp=10

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