A New Configuration of Two-Wheeled Inverted Pendulum: A Lagrangian-Based Mathematical Approach

K M Goher [1] and M O Tokhi [2]
[1]Department of Mechanical and Industrial Engineering
College of Engineering, Sultan Qaboos University, Oman, e-mail: kgoher@squ.edu.om
[2]Department of Automatic Control and Systems Engineering,
The University of Sheffield, United Kingdom

***Abstract*— This work presents a novel design of two-wheeled vehicles. The proposed design provides the vehicle with more flexibility in terms of the increased degrees of freedom which enable the vehicle to enlarge its working space. The additional translational degree of freedom (DOF), offered by the linear actuator, assists any attached payload to reach higher levels as and when required. The model of the system mimics the scenario of double inverted pendulum on a moving base. However, it is further complicated due to the addition of a one more (DOF). As adding more degrees of freedom to the system increases the degree of complexity, Lagrangian dynamic formulation is used, due to its relative simplicity, to derive the system dynamics. The new developed configurations is of great importance in various applications including self balance robots, wheelchairs on two wheels, stability analysis of multi segment gaits and multi links cranes etc. In order to maintain the system nonlinear characteristics, the system model is derived with the consideration of the joints friction based on the Coulomb friction model. An investigation is carried out on the impact of the joints damping on the stability of the system.**

Keywords- Lagrangian formulation, modelling and simulation , double inverted pendulum.

I. INTRODUCTION

Inverted pendula are currently used as teaching aids and research experiments. Quanser (2004), a supplier of educational and research based equipment produce modular systems which can be configured as single or double inverted pendula. Their range offers both a rotary and a linear version. Many researchers have also built their own inverted pendulum systems (Åström and Furuta, 1996; Brockett & Hongyi, 2003; Rubi, 2002) to suit their investigations.

Theoretically, any number of links can be mounted on the cart or rotor and successfully held in the all-up configuration (Cazzolato, 2004). The most reported to have been successfully balanced is four (Googol, 2004). Video footage of this feat can be viewed at the Googol website. Three link systems (triple inverted linear pendulum) have been observed as demonstrated by the Max Plank Institute (2004) and Quanser (2004). For all of these systems, each link (including the rotary link or cart) has only one degree of freedom.

The concept of balancing robot is based on the inverted pendulum model. This model has been widely used by researches around the world, [1], [3], [4] and [5], in controlling a system not only in designing wheeled robot but also other types of robots as well as legged robots. The inverted pendulum problem is common in the field of control engineering. The uniqueness and wide application of technology derived from this unstable system has drawn interest of many researches, [2], [6], [8] and [9], and robotics enthusiasts around the world. In recent years, researchers have applied the idea of a mobile inverted pendulum model to various applications including the design of walking gaits for humanoid robots, robotic wheelchairs [7] and personal transport systems [10].

In this paper, due to the highly nonlinearity characteristics of the multi-inputs multi-outputs (MIMO) two-wheeled system, mathematical model using Lagrangian approach has been developed considering all possible system parameters maintaining system nonlinearities and complexity. Special concern is paid to the damping characteristics of the joints as well as the dynamics of the overall centre of mass (COM) of the system. Further investigation will focus on the impact of the lower parts inertia and size; wheels mass and size, counter weights to maintain balance and overall dimensions of lower parts of the vehicle.

II. SYSTEM DESCRIPTION

The vehicle considered in this work, shown in Figure 1, comprises a rod on an axle incorporating two wheels as described in Figures 1. The intermediate body (IB) of the vehicle encompasses two segments; link 1 and link 2, where the second link is a set of two coaxial-parts connected by a linear drive to actuate the upper part and the attached payload. The vehicle is driven by four driving direct motors (DC); two motors drive the vehicle wheels and in turn the entire system, a motor driving link 2 and a linear actuator between the two parts of link 2. The two wheels driving motors along with the motor driving link 2 help the system to be stabilized at the up right position or at any angular position as required by the system control strategy. Using the linear actuator increases the system degrees of freedom by allowing the second link to extend the second to achieve further levels of height.

The angular positions of link 1 and link 2; θ_1 and θ_2, are measured from the positive vertical Z axis. The linear displacement of the payload; Q, is measured, from position O_2, along link 2. In order to achieve a certain angular position from the upright vertical axis, the vehicle is linearly moved in the XY plane undergoing a planar motion in both directions. The motion of the vehicle in the XY plane relies, generally, on the degree of actuation of the wheels driving motors and hence on the control signals in correspondence to the pre-assigned angular position of the IB.

Upon receiving the corresponding signal from the controllers, the wheels start to respond, independently, by rotating with the appropriate speed and in a direction relative to the nature of the received signals to the corresponding controller. The linear actuator is working to activate the upper part of link 2 and the attached payload to extend in accordance to the position measurement, as required, of the payload.

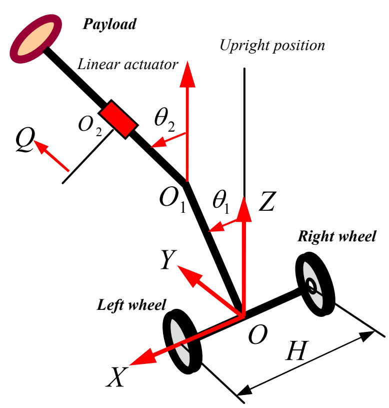

Fig. 1 Two-wheeled vehicle with an extended rod

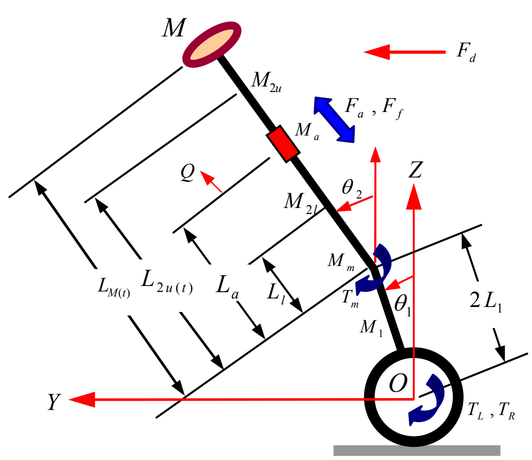

Fig. 2 Positions of vehicle main parts and *Com*

III. LAGRANGIAN MODELING APPROACH

Application of the Euler-Lagrange equations leads to a set of coupled second-order ordinary differential equations and provides a formulation of the dynamic equations of motion equivalent to those derived using Newton's formulation. However, the Lagrangian approach is advantageous for complex systems such as multi-link systems.

TABLE I. PARAMETERS LABEL AND DESCRIPTION

Terminology	***Description***	***Units***
$L_{M(t)}$	*Distance to the COM of the payload*	*m*
$L_{2u(t)}$	*Distance to the COM of the upper part of link*	*m*
L_a	*Position of the linear actuator*	*m*
L_l	*Half length of link 1*	*m*
$2L_l$	*Length of link 1*	*m*
H	*Distance between wheels, along X axis*	*m*
Q	*Displacement of the linear actuator*	*m*
M_l	*Mass of link 1*	*kg*
M_m	*Mass of motor driving link 2*	*kg*
M_{2l}	*Mass of the lower part of link 2*	*kg*
M_a	*Mass of the linear actuator*	*kg*
M_{2u}	*Mass of the upper part of link 2*	*kg*
M	*Payload mass*	*kg*
T_R, T_L	*Wheels driving torques*	*kg*
T_m	*Motor torque*	*N.m*
F_a	*Linear actuator force*	*N*
F_f	*Frictional force in the linear actuator*	*N*
F_d	*External disturbance force*	*N*
θ_1	*Angular position of link 1 to the positive Z axis*	*rad*
θ_2	*Angular position of link 2 to the positive Z axis*	*rad*

There are several reasons for using Lagrangian approach. First, the derivation is based on energy calculations of the physical system. As energy calculations are independent of vectors representation, the derivation is simple in compared to Newton-Euler formulation. Calculation of the system energy is of great importance for power consumptions by the system and designing the appropriate actuators to derive the system.

Second, using generalized coordinates to describe the system degrees of freedom simplifies and describes more naturally the situation using more sensible coordinates. Another advantage of Lagrange's dynamic is the method of Lagrangian multipliers. That method is usable when you do not know the nature of some force during constrained motion. Lagrangian Dynamics is derived from Newton's Laws and so has the same restrictions as those laws.

Using Lagrange dynamic formulation for the system dynamics, the following dynamic equation can be expressed for an *n* degrees of freedom (DOF) system

$$\frac{dq}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \qquad (1)$$

Where

$L = T - V$, is the Lagrangian function,

$Q_i =$ Generalized force associated with a generalized coordinate q_i

$q_i =$ Generalized coordinate,

$n =$ Number of degrees of freedom of the system,

$T =$ System kinetic energy, and
$V =$ System potential energy

A. System Energy Requirements

Since Lagrangian technique consider the system energy, consisting of Kinetic and Potential energy, thus the total energy, U of the two-wheeled wheelchair can be described as the sum of the kinetic energy, T , and potential energy, V , of the system components; wheel, lower and upper parts of links and the payload as:

$$U = T + V \tag{2}$$

$$T = T_c + T_{v,\varphi} + T_1 + T_m + T_{2l} + T_a + T_{2u} + T_M \tag{3}$$

$$V = V_1 + V_m + V_{2l} + V_a + V_{2u} + V_M \tag{4}$$

$$T_c = \left(M_W R_W^2 + J_W\right)\left(\dot{\delta}_L^2 + \dot{\delta}_R^2\right) \tag{5}$$

$$T_{v,\varphi} = \frac{1}{2}\left(2J_W + J_{IB}\right)\dot{\phi}^2 \tag{6}$$

where

$$M_c = 2M_w + M_{axle} + 2M_{driving\ motors} + 2M_{gearbox}$$

The pendulum kinetic energy can be expressed as the sum of its translational energy and rotational energy;

$$T_1 = \frac{1}{2}M_1\left(\left(\frac{R_W}{2}\left(\dot{\delta}_L + \dot{\delta}_R\right) + L_1\dot{\theta}_1\cos\theta_1\right)^2 + \left(L_1\dot{\theta}_1\sin\theta_1\right)^2\right) + \frac{1}{2}J_1\dot{\theta}_1^2 \tag{7}$$

$$T_m = \frac{1}{2}M_m\left(\left(\frac{R_W}{2}\left(\dot{\delta}_L + \dot{\delta}_R\right) + 2L_1\dot{\theta}_1\cos\theta_1\right)^2 + \left(2L_1\dot{\theta}_1\sin\theta_1\right)^2\right) + \frac{1}{2}J_m\dot{\theta}_1^2 \tag{8}$$

$$T_{2l} = \frac{1}{2}M_{2l}\begin{pmatrix}\left(\frac{R_W}{2}\left(\dot{\delta}_L + \dot{\delta}_R\right) + 2L_1\dot{\theta}_1\cos\theta_1 + L_{2l}\dot{\theta}_2\cos\theta_2\right)^2 \\ + \left(2L_1\dot{\theta}_1\sin\theta_1 + L_{2l}\dot{\theta}_2\sin\theta_2\right)^2\end{pmatrix} + \frac{1}{2}J_{2L}\dot{\theta}_2^2 \tag{9}$$

$$T_a = \frac{1}{2}M_a\begin{pmatrix}\left(\frac{R_W}{2}\left(\dot{\delta}_L + \dot{\delta}_R\right) + 2L_1\dot{\theta}_1\cos\theta_1 + 2L_{2l}\dot{\theta}_2\cos\theta_2\right)^2 \\ + \left(2L_1\dot{\theta}_1\sin\theta_1 + 2L_{2l}\dot{\theta}_2\sin\theta_2\right)^2\end{pmatrix} + \frac{1}{2}J_a\dot{\theta}_2^2 \tag{10}$$

$$T_{2u} = \frac{1}{2}M_{2u}\begin{pmatrix}\dot{Q}^2 + \left(\frac{R_W}{2}\left(\dot{\delta}_L + \dot{\delta}_R\right) + 2L_1\dot{\theta}_1\cos\theta_1 + L_{2u(t)}\dot{\theta}_2\cos\theta_2\right)^2 \\ + \left(2L_1\dot{\theta}_1\sin\theta_1 + L_{2u(t)}\dot{\theta}_2\sin\theta_2\right)^2\end{pmatrix} + \frac{1}{2}J_{2u}\dot{\theta}_2^2 \tag{11}$$

$$T_M = \frac{1}{2}M\begin{pmatrix}\dot{Q}^2 + \left(\frac{R_W}{2}\left(\dot{\delta}_L + \dot{\delta}_R\right) + 2L_1\dot{\theta}_1\cos\theta_1 + L_{M(t)}\dot{\theta}_2\cos\theta_2\right)^2 \\ + \left(2L_1\dot{\theta}_1\sin\theta_1 + L_{M(t)}\dot{\theta}_2\sin\theta_2\right)^2\end{pmatrix} + \frac{1}{2}J_M\dot{\theta}_2^2 \tag{12}$$

where J_l, J_a, J_u and J_M are the mass moments of inertia of the lower rod, linear actuator, upper rod and the payload respectively around the IB centre of mass.

Since there is no motion for the vehicle in the Z direction as the wheels remain in full contact with the ground; $\ddot{Z}_R = \ddot{Z}_L = \ddot{Z}_O = 0$, there is no potential energy for the cart in the Z direction. The potential energy of various components can be expressed as:

$$V_1 = M_1 g L_1 \cos\theta_1 \tag{13}$$

$$V_m = M_m g\left(2L_1\right)\cos\theta_1 \tag{14}$$

$$V_{2l} = M_{2l} g\left(2L_1\cos\theta_1 + L_{2l}\cos_2\right) \tag{15}$$

$$V_a = M_a g\left(2L_1\cos\theta_1 + 2L_{2l}\cos_2\right) \tag{16}$$

$$V_{2u} = M_{2u} g\left(2L_1\cos\theta_1 + 2L_{2u(t)}\cos_2\right) \tag{17}$$

$$V_M = Mg\left(2L_1\cos\theta_1 + 2L_{M(t)}\cos_2\right) \tag{18}$$

Where $L_{2u(t)}$ and $L_{M(t)}$ are the positions of the centre of mass of upper part of link 2 and the payload respectively. Both variables are time dependent, in correspondence to the displacement caused by the linear actuator, and can be expressed as:

$$L_{2u(t)} = 2L_{2l} + L_{2u} + Q \tag{19}$$

$$L_{M(t)} = 2L_{2l} + 2L_{2u} + Q \tag{20}$$

Manipulating the above equations yield the following two expressions for the total kinetic and total potential energies respectively of the system;

$$\begin{aligned} T_v = {} & C_{21}\left(\dot{\delta}_L^2 + \dot{\delta}_R^2\right) + C_{22}\dot{\delta}_L\dot{\delta}_R + \frac{1}{2}C_8\dot{Q}^2 + \frac{1}{2}C_{16}\dot{\phi}_2 + C_{18}\dot{\theta}_1^2 \\ & + \frac{1}{2}\left(C_{20} + C_{12}Q + C_8Q^2\right)\dot{\theta}_2^2 + C_9\frac{R_W}{2}L_1\dot{\theta}_1\left(\dot{\delta}_L + \dot{\delta}_R\right)\cos\theta_1 \\ & + \frac{R_W}{2}\left(C_{10} + C_8Q\right)\dot{\theta}_2\left(\dot{\delta}_L + \dot{\delta}_R\right)\cos\theta_2 + 2L_1\left(C_{10} + C_8Q\right)\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) \end{aligned} \tag{20}$$

$$V_v = C_3\, g\cos\theta_1 + \left(C_{15} + C_8Q\right)g\cos\theta_2 \tag{21}$$

IV. VEHICLE DYNAMICS

The Lagrangian equation of motion is represented as,

$$\frac{dq}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \tag{22}$$

where q_i represents a particular generalized coordinate, and

$$\dot{q}_i = \frac{dq_i}{dt} \tag{23}$$

The overall motion of the system can be described using Eq. (3) according to separate each generalized coordinate in a system. For example, in this case, there are five generalized coordinates taking part in the system motion, thus, the generalized coordinates of the system are chosen as

$$q_i = \left[\delta_{\mathrm{L}} \quad \delta_{\mathrm{R}} \quad \theta_1 \quad \theta_2 \quad \mathrm{Q}\right]^T \tag{24}$$

The generalized force is expressed as,

$$Q_i = \left[T_{LT} \quad T_{RT} \quad 0 \quad T_{MT} \quad F_{aT}\right]^T \tag{25}$$

Where the generalized forces and moments are expressed as follows:

$$T_{LT} = T_L - T_{fL} \tag{26}$$

$$T_{RT} = T_R - T_{fR} \tag{27}$$

$$T_{MT} = T_M - T_{fM} \tag{28}$$

$$F_{aT} = F_a - F_{fa} \tag{29}$$

A. Joints friction effects

Based on the Coulomb's friction model and assuming the same frictional coefficients at all the joints, the frictional moments and forces are expressed as the following:

$$T_{fL} = c_v \dot{\delta}_L + c_c \sin \dot{\delta}_L \tag{30}$$

$$T_{fR} = c_v \dot{\delta}_R + c_c \sin \dot{\delta}_R \tag{31}$$

$$T_{fM} = c_v \dot{\theta}_2 + c_c \sin \dot{\theta}_2 \tag{32}$$

$$F_{fa} = c_v \dot{Q} + c_c \sin \dot{Q} \tag{33}$$

Where T_{fL} , T_{fR} and T_{fM} are the frictional moments at the left and right wheels joints and the intermediate joint between link 1 and link 2 respectively and F_{fa} is the frictional force exist in the linear actuator. c_v and c_c are the viscous and Coulomb friction coefficients at the vehicle joints respectively. $\dot{\delta}_L$ and $\dot{\delta}_R$ are the rate of angular rotations at the left and right wheels respectively. $\dot{\theta}_2$ is the rate of the angular position of link 2 and $\dot{Q}$ is the velocity of the attached payload.

B. Lagrangian formulation

The Lagrangian function of the system; L can be expressed as the difference between the system kinetic and potential energy as the following:

$$\begin{aligned}
L &= C_{21}\left(\dot{\delta}_L^2 + \dot{\delta}_R^2\right) + C_{22}\dot{\delta}_L\dot{\delta}_R + \frac{1}{2}C_8\dot{Q}^2 + \frac{1}{2}C_{16}\dot{\phi}_2 + C_{18}\dot{\theta}_1^2 \\
&+ \frac{1}{2}\left(C_{20} + C_{12}Q + C_8Q^2\right)\dot{\theta}_2^2 + C_9\frac{R_W}{2}L_1\dot{\theta}_1\left(\dot{\delta}_L + \dot{\delta}_R\right)\cos\theta_1 \\
&+ \frac{R_W}{2}\left(C_{10} + C_8Q\right)\dot{\theta}_2\left(\dot{\delta}_L + \dot{\delta}_R\right)\cos\theta_2 + 2L_1\left(C_{10} + C_8Q\right)\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) \\
&- C_3 g\cos\theta_1 - \left(C_{15} + C_8Q\right)g\cos\theta_2
\end{aligned} \tag{34}$$

The Lagrangian equations of motion of the vehicle can be represented as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\delta}_L}\right) - \frac{\partial L}{\partial \delta_L} = T_L - T_{fL} \tag{35}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\delta}_R}\right) - \frac{\partial L}{\partial \delta_R} = T_R - T_{fR} \tag{36}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \tag{37}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = T_M - T_{fM} - L_d F_d \tag{38}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\right) - \frac{\partial L}{\partial Q} = F_a - F_{fa} \tag{39}$$

C. Vehicle dynamic equations

The system is described by a set of Manipulating the above expressions yields the following five highly non-linear differential equations describing the vehicle dynamics alongside the driving moments and an external disturbance force as:

$$\begin{aligned}
&2C_{21}\ddot{\delta}_L + C_{22}\ddot{\delta}_R - C_9\frac{R_W}{2}L_1\dot{\theta}_1^2\sin\theta_1 + C_9\frac{R_W}{2}\ddot{\theta}_1\cos\theta_1 \\
&- \frac{R_W}{2}C_{10}\dot{\theta}_2^2\sin\theta_2 + \frac{R_W}{2}C_{10}\ddot{\theta}_2\cos\theta_2 + \frac{R_W}{2}C_8\dot{Q}\dot{\theta}_2 + Q\ddot{\theta}_2 \\
&+ \dot{Q}\cos\theta_2 - Q\dot{\theta}_2^2\sin\theta_2 + \ddot{\theta}_2\cos\theta_2 - \dot{\theta}_2^2\sin\theta_2 = T_L - T_{fL}
\end{aligned} \tag{40}$$

$$\begin{aligned}
&2C_{21}\ddot{\delta}_R + C_{22}\ddot{\delta}_L - C_9\frac{R_W}{2}L_1\dot{\theta}_1^2\sin\theta_1 + C_9\frac{R_W}{2}\ddot{\theta}_1\cos\theta_1 \\
&- \frac{R_W}{2}C_{10}\dot{\theta}_2^2\sin\theta_2 + \frac{R_W}{2}C_{10}\ddot{\theta}_2\cos\theta_2 + \frac{R_W}{2}C_8\dot{Q}\dot{\theta}_2 + Q\ddot{\theta}_2 \\
&+ \dot{Q}\cos\theta_2 - Q\dot{\theta}_2^2\sin\theta_2 + \ddot{\theta}_2\cos\theta_2 - \dot{\theta}_2^2\sin\theta_2 = T_R - T_{fR}
\end{aligned} \tag{41}$$

$$\begin{aligned}
&2C_{18}\ddot{\theta}_1 + C_{22}\ddot{\delta}_L + C_9\frac{R_W}{2}L_1\left(\ddot{\delta}_L + \ddot{\delta}_R\right)\cos\theta_1 \\
&- C_9\frac{R_W}{2}L_1\dot{\theta}_1\left(\dot{\delta}_L + \dot{\delta}_R\right)\sin\theta_2 - 2L_1\left(\left(C_8 + C_{10}\right)\left(\dot{\theta}_2 - \dot{\theta}_1\right)\sin\left(\theta_1 - \theta_2\right)\right. \\
&+ 2C_8\ddot{Q}\dot{\theta}_2 + +2C_8Q\ddot{\theta}_2 + \left(C_8\ddot{Q} + \left(C_8 + C_{10}\right)\ddot{\theta}_2\right)\left(\sin\theta_1 + \sin\theta_2 + \cos\theta_1 + \cos\theta_2\right) \\
&+ \left(C_8Q + \left(C_8 + C_{10}\right)\dot{\theta}_2\right)\left(\dot{\theta}_1\left(\cos_1 - \sin\theta_1\right) + \dot{\theta}_2\left(\cos\theta_2 - \cos\theta_1\right)\right)\Big) \\
&+ C_9\frac{R_W}{2}L_1\dot{\theta}_1\left(\dot{\delta}_L + \dot{\delta}_R\right)\sin\theta_1 + 2L_1\left(C_{10} + C_8Q\right)\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_1 - \theta_2\right) - C_3 g\sin\theta_1 = 0
\end{aligned} \tag{42}$$

$$\begin{aligned}
&\left(\left(C_{20} + C_{12}Q + C_8Q^2\right) + 2L_1\left(C_8 + C_{10}\right)\left(\sin\theta_1 + \sin\theta_2 + \cos\theta_1 + \cos\theta_2\right)\right)\ddot{\theta}_2 \\
&+ \left(\left(C_{12}\dot{Q} + 2C_8Q\dot{Q}\right) + \frac{R_W}{2}C_8\left(Q - 1\right)\left(\dot{\delta}_L + \dot{\delta}_R\right)\sin\theta_2\right)\dot{\theta}_2 \\
&+ \frac{R_W}{2}\left(C_8Q + \left(C_8 + C_{10}\right)\cos\theta_2\right)\left(\ddot{\delta}_L + \ddot{\delta}_R\right) + \frac{R_W}{2}C_8\dot{Q}\left(\dot{\delta}_L + \dot{\delta}_R\right) \\
&+ 2L_1C_8\left(2\dot{\theta}_2\dot{Q} + 2Q\ddot{\theta}_2\right) + 2L_1\dot{\theta}_1\left(\left(C_8 + C_{10}\right)\dot{\theta}_2 + C_8Q\right)\left(\cos\theta_1 - \sin\theta_1\right) \\
&+ 2L_1\dot{\theta}_2\left(C_{10}\dot{\theta}_2 + C_8\left(\dot{\theta}_2 + Q\right)\right)\left(\cos\theta_2 - \sin\theta_2\right) \\
&+ \left(C_8\frac{R_W}{2}Q\dot{\theta}_2 + \left(C_{15} + C_8Q\right)g\right)\sin\theta_2 + C_8\dot{Q}\cos\theta_2 \\
&+ 2L_1\left(C_8 + C_{10}\right)\sin\left(\theta_1 - \theta_2\right) - \left(C_{10} + C_8Q\right)\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_1 - \theta_2\right) \\
&= T_m - T_{fm} - L_d F_d
\end{aligned} \tag{43}$$

$$\begin{aligned}
&C_8\ddot{Q} - \frac{1}{2}\left(C_{12} + 2C_8Q\right)\dot{\theta}_2^2 - \frac{R_W}{2}C_8\dot{\theta}_2\left(\dot{\delta}_L + \dot{\delta}_R\right)\cos\theta_2 \\
&- 2L_1C_8\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right) + C_8 g\cos\theta_2 = F_a - F_{fa}
\end{aligned} \tag{44}$$

D. Intermediate body centre of mass (COM)

Calculating of the IB centre of mass is of great importance, the following equations describe how the *COM* of the intermediate body is calculated. Considering the moments of the vehicle main parts around the position of the *COM* in two mutually perpendicular directions; X and Z yields the following expressions:

$$L_g M_{eq}\sin\alpha = \left(M_1 L_1 + M_m(2L_1) + 2L_1(M_{2l} + M_a + M_{2u} + M)\right)\sin\theta_1 + \left(L_{2l}M_{2l} + 2L_{2l}M_a + M_{2u}L_{2u(t)} + ML_{M(t)}\right)\sin\theta_2 \quad (45)$$

$$L_g M_{eq}\cos\alpha = \left(M_1 L_1 + M_m(2L_1) + 2L_1(M_{2l} + M_a + M_{2u} + M)\right)\cos\theta_1 + \left(L_{2l}M_{2l} + 2L_{2l}M_a + M_{2u}L_{2u(t)} + ML_{M(t)}\right)\cos\theta_2 \quad (46)$$

where L_g is a two dimensional vector in the XZ plane with a length represents the position of centre of mass (*COM*) of the intermediate body, and it is a time-dependent varying with the angular positions of link 1 and link 2 and the linear displacement of the attached payload and can be calculated using;

$$L_g = \frac{C_{13}\sin\theta_1 + (C_{14} + C_8 Q)\sin\theta_2}{M_{eq}\sin\alpha} \quad (47)$$

Where $M_{eq} = M_1 + M_m + M_{2l} + M_a + M_{2u} + M$

Manipulating the above equations yields the calculation of the angular position of *COM* in the XZ plane as follows:

$$\alpha = \arctan\frac{C_{13}\sin\theta_1 + (C_{14} + C_8 Q)\sin\theta_2}{C_{13}\cos\theta_1 + (C_{14} + C_8 Q)\cos\theta_2} \quad (48)$$

The equation of motions derived were constructed in the Simulink environment for testing with conventional PID controller.

V. CONTROL STRATEGY

The strategy to control the system depends on developing a feedback control strategy of five control loops as shown in Figure 3. In order to drive the vehicle to undergo a specific planar motion in the XY plane, two feedback loops are developed. The input to the both control loops is the error in the angular position of each wheel which measures the difference between the desired and actual angular position of the corresponding wheel. The angular position of the intermediate body is controlled by the measurement of the error in the position of link 1 and link 2. In order to control the position of the attached payload, a feedback control loop is developed with the error in the payload position as an input and the actuation force as the output of the control loop.

The system of vehicle considered in this paper is a multi input multi output (MIMO) system characterized by its high nonlinearity and the highly coupled dynamics. The inputs to the system are the main motors driving torques; T_L and T_r, torque driving the motor activating link 2; T_m, the linear actuator force; F_a and an external disturbance force; F_d. The system posses five outputs; the angular positions of the left and right wheels; δ_L and δ_R respectively, the angular positions of the IB segments; θ_1 and θ_2 and the displacement of the payload; Q.

The developed control strategy is implemented on the system model with the full results to be presented and discussed in the full paper.

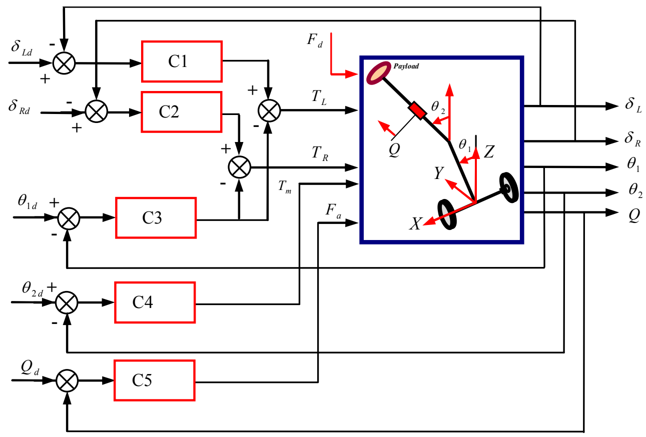

Figure 1. Schematic description of the control strategy

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